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MASS MINIMIZATION FOR A SPHERICAL SCREEN WITH A SPECIFIED LEVEL OF TRANSMITTED WAVE ENERGY

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A layered spherical screen with minimum weight exposed to a spherical wave is synthesized from a finite set of elastic homogeneous isotropic materials under constraints imposed on the wave energy transferred through the screen and the screen thickness. The necessary optimality conditions are obtained and an example of calculation of the optimal structure is given.

1. Formulation of the Problem. Problems of the optimal design of plane layered structures subjected to wave actions are dealt with in a number of papers (see, e.g., [1–3]). We consider the following problem. Let W be a set of k homogeneous isotropic materials. From the given set, it is required to synthesize a layered spherical screen with minimum weight under specified constraints on the wave energy and thickness of the screen.

In the case of central symmetry, the stress–strain state of a layered medium is described in the spherical coordinates (r, θ, φ) by the equation of motion

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2}{r}\,(\sigma_{rr} - \sigma_{\varphi\varphi}) = \rho\,\frac{\partial^2 u_r}{\partial t^2} \tag{1.1}$$

and the Hooke's law

$$\sigma_{rr} = \rho c_l^2\,\frac{\partial u_r}{\partial r} + 2\rho(c_l^2 - 2c_t^2)\,\frac{u_r}{r}, \quad \sigma_{\varphi\varphi} = \rho(c_l^2 - 2c_t^2)\frac{\partial u_r}{\partial r} + 2\rho(c_l^2 - c_t^2)\,\frac{u_r}{r}. \tag{1.2}$$

Here $u_r(r,t)$ is the radial displacement, $\sigma_{rr}(r,t)$ and $\sigma_{\varphi\varphi}(r,t)$ are the radial and circumferential stresses, respectively, $c_l(r)$ and $c_t(r)$ are parameters of the medium that are expressed in terms of Young's modulus $E(r)$, Poisson's ratio $\nu(r)$, and the density $\rho(r)$ of the layer materials by the formulas

$$c_l^2 = \frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}, \qquad c_t^2 = \frac{E}{2\rho(1+\nu)}.$$

Let r_1 and r_2 be the inner and outer radii of the layered screen, respectively. From a point source located in the coordinate origin, a spherical wave containing the entire frequency spectrum is incident on the internal boundary of the screen r_1 (see Fig. 1).

For media whose parameters are independent of time, one can use a spectral representation with respect to time [4] to reduce the number of independent variables in Eq. (1.1) and formulate the initial problem in terms of the spectral densities of the radial velocity $v_r(r,\omega) = -i\omega u_r(r,\omega)$ and the radial stress $\sigma_{rr}(r,\omega)$.

On the external and internal boundaries of the screen layers $r_i \in [r_1, r_2]$, where the acoustic properties of the layer materials and ambient medium undergo jumps, it is necessary to specify the following conjugation conditions [continuity of the velocity $v_r(r,\omega)$ and stress $\sigma_{rr}(r,\omega)$]:

$$[v_r(r_i,\omega)]\Big|_-^+ = [\sigma_{rr}(r_i,\omega)]\Big|_-^+ = 0. \tag{1.3}$$

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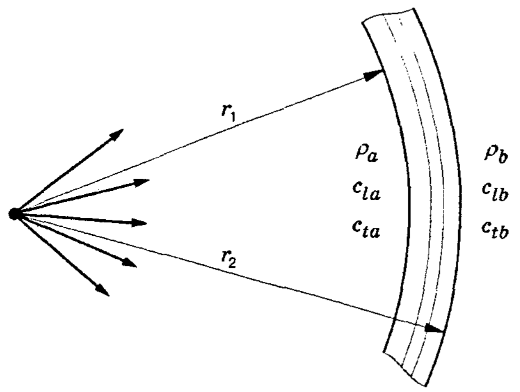

Fig. 1

We derive the boundary conditions for system (1.1), (1.2). For convenience, we consider this system not on the entire r axis but on the segment $[r_1, r_2]$ occupied by the layered screen. The effect of the regions $r < r_1$ and $r > r_2$ is taken into account by the boundary conditions for $r = r_1$ and $r = r_2$.

We consider the internal region $r < r_1$ occupied by a medium with acoustic properties ρ_a, c_{la}, and c_{ta}. Let a monochromatic spherical wave with the potential

$$\Phi_1 = [b_1(\omega)/r] \exp[i\omega(r/c_{la} - t)]$$

emanate from a source located at the coordinate origin and be incident on the screen. A reflected wave with the potential

$$\Phi_1 = [b_1(\omega)/r] \exp[i\omega(r/c_{la} - t)]$$

occurs in the region $r < r_1$. In this case, the solution of system (1.1), (1.2) on the boundary $r = r_1$ can be written as

$$v_r(r_1, \omega) = b_1(\omega)\Big(\frac{i\omega}{r_1^2} + \frac{\omega^2}{r_1 c_{la}}\Big) \exp\Big(\frac{i\omega r_1}{c_{la}}\Big) + b_2(\omega)\Big(\frac{i\omega}{r_1^2} - \frac{\omega^2}{r_1 c_{la}}\Big) \exp\Big(-\frac{i\omega r_1}{c_{la}}\Big),$$

$$\sigma_{rr}(r_1, \omega) = b_1(\omega)\Big[4\rho_a c_{ta}^2\Big(\frac{1}{r_1^3} - \frac{i\omega}{r_1^2 c_{la}}\Big) - \rho_a \frac{\omega^2}{r_1}\Big] \exp\Big(\frac{i\omega r_1}{c_{la}}\Big) \tag{1.4}$$

$$+ b_2(\omega)\Big[4\rho_a c_{ta}^2\Big(\frac{1}{r_1^3} + \frac{i\omega}{r_1^2 c_{la}}\Big) - \rho_a \frac{\omega^2}{r_1}\Big] \exp\Big(-\frac{i\omega r_1}{c_{la}}\Big).$$

The amplitude $b_1(\omega)$ of the incident-wave potential is expressed in terms of the parameters of this wave. The unknown amplitude $b_2(\omega)$ of the reflected wave can be eliminated from (1.4). After simple manipulations, we obtain

$$g_{11} v_r(r_1, \omega) + g_{12} \sigma_{rr}(r_1, \omega) = g_{13}, \tag{1.5}$$

where $g_{11} = \rho_a c_{la} \omega^2 r_1^2 - 4\rho_a c_{ta}^2 (c_{la} + i\omega r_1)$, $g_{12} = i\omega r_1 c_{la} - \omega^2 r_1^2$, and $g_{13} = 2b_1(\omega)\rho_a r_1 \omega^4 \exp(i\omega r_1/c_{la})$.

The coefficients g_{ij} in (1.5) depend on the properties of the medium that occupies the region $r < r_1$ and on the incident-wave parameters.

The external region $r > r_2$ occupied by a medium with the acoustic properties ρ_b, c_{lb}, and c_{tb} is considered in a similar manner. In this region, just one transmitted refracted wave propagates. Eliminating the unknown amplitude of the refracted-wave potential from the solution of system (1.1), (1.2) in the region $r > r_2$, we obtain the following boundary condition on the boundary $r = r_2$:

$$g_{21} v_r(r_2, \omega) + g_{22} \sigma_{rr}(r_2, \omega) = 0. \tag{1.6}$$

Here $g_{21} = \rho_b c_{lb} \omega^2 r_2^2 - 4\rho_b c_{tb}^2 (c_{lb} - i\omega r_2)$ and $g_{22} = i\omega r_2 c_{lb} + \omega^2 r_2^2$.

Thus, in the case of the incident spherical wave, the parameters of the wave process, i.e., the velocity and stress distributions in the layered screen that occupies the region $[r_1, r_2]$, are determined from the solution of the boundary-value problem (1.1)–(1.3), (1.5), (1.6).

We use the change of coordinates

$$r = r_1 + x\,(r_2 - r_1) \qquad (x \in [0,\,1]), \tag{1.7}$$

which maps the region of definition $[r_1,\, r_2]$ onto the interval [0, 1]. We introduce the piecewise-constant function

$$\alpha(x) = \{\alpha_j\colon x \in [x_j,\, x_{j+1}),\, j = 1, \ldots, n\}, \quad x_1 = 0, \quad x_{n+1} = 1, \tag{1.8}$$

which characterizes the structure of the layered screen: the number, dimensions, and materials of the constituent layers. The values of α_j belong to the discrete finite set

$$U = \{1, 2, \ldots, k\}, \tag{1.9}$$

which corresponds to the given set of materials W. All characteristics of the materials from the set W are functions of the distribution $\alpha(x)$ on the segment [0, 1].

Since the structure of the layered screen is determined by the function $\alpha(x)$ and the thickness is determined by its dimensions r_1 and r_2, we consider the pair $\{\alpha(x), r_2\}$ as the control (without loss of generality, the inner radius r_1 is assumed to be fixed), where $\alpha(x) \in U$ (1.9) and

$$0 < r_1 < a \leqslant r_2 \leqslant b \tag{1.10}$$

(a and b are the specified limits within which the outer radius r_2 can be varied).

The problem of the optimal design of a spherical screen is formulated as follows. From the piecewise-constant functions $\alpha(x)$ with values of U and the parameters r_2 in the range $[a,\, b]$ that satisfy inequality (1.10), we need to find a control $\{\alpha(x), r_2\}$ that provides for a minimum of the weight functional

$$F_0(\alpha, r_2) = 4\pi \int\limits_{r_1}^{r_2} \rho(\alpha) r^2\, dr = \int\limits_0^1 G(\alpha, r_2, x)\, dx \tag{1.11}$$

with the specified constraint on the wave energy

$$F_1(\alpha, r_2, v_r, \sigma_{rr}) = J(\alpha, r_2, v_r, \sigma_{rr}) - \eta J_0 \leqslant 0. \tag{1.12}$$

Here $J(\alpha, r_2, v_r, \sigma_{rr})$ is the time-averaged wave-energy flux transferred by the spherical wave through the surface $r = r_2$ in the r axis, J_0 is the average energy flux in the incident wave, and η is the energy transmission factor of the screen [4], i.e., the fraction of the energy flux in the incident wave that can pass into the region $r > r_2$.

Using the expression for the energy flux [5] and bearing (1.6) in mind, we write the functional $J(\alpha, r_2, v_r, \sigma_{rr})$ in the form

$$J(\alpha, r_2, v_r, \sigma_{rr}) = -2\pi r_2^2 \int\limits_0^\infty \mathrm{Re}\,(\bar{v}_r(r_2, \omega)\sigma_{rr}(r_2, \omega))\, d\omega = 2\pi r_2^2 \int\limits_0^\infty |\sigma_{rr}(r_2, \omega)|^2 \mathrm{Re}\Big(\frac{g_{22}}{g_{21}}\Big)\, d\omega$$

(the bar above the function denotes complex conjugation). The corresponding expression for the energy flux J_0 in the incident wave takes the form

$$J_0 = -2\pi r_1^2 \int\limits_0^\infty \mathrm{Re}\,(\bar{v}_r(r_1, \omega)\sigma_{rr}(r_1, \omega))\, d\omega = \frac{2\pi\rho_a}{c_{la}} \int\limits_0^\infty \omega^4 |b_1(\omega)|^2\, d\omega,$$

where $b_1(\omega)$ is the amplitude of the incident spherical wave potential.

2. Necessary Optimality Conditions. To obtain the necessary optimality conditions in problem (1.1)–(1.12), it is necessary to express variations in the objective function (1.11) and constraint (1.12) in terms of variation in the control $\{\alpha(x), r_2\}$. To this end, we transform the boundary-value problem (1.1)–(1.3), (1.5), (1.6).

The conjugation conditions (1.3) and relation (1.7) allow one to introduce the phase variables

$$\boldsymbol{Y}(x,\omega) = (y_1, y_2)^{\mathrm{t}} = (v_r, \sigma_{rr})^{\mathrm{t}}$$

that are continuous on the segment [0, 1] (the superscript "t" denotes transposition of the corresponding vector or matrix).

In the new variables, the controlled system (1.1)–(1.3), (1.5), (1.6) becomes

$$\begin{gathered}\boldsymbol{Y}'(x,\omega) = A(\alpha, r_2, x, \omega)\boldsymbol{Y}(x,\omega),\\ g_{11}y_1(0,\omega) + g_{12}y_2(0,\omega) = g_{13}, \qquad g_{21}y_1(1,\omega) + g_{22}y_2(1,\omega) = 0,\end{gathered} \tag{2.1}$$

where the prime denotes differentiation with respect to the x coordinate, the coefficients g_{ij} are given by relations (1.5) and (1.6), and the components of the matrix A have the form

$$a_{11} = \frac{1}{r}\Big(4\frac{c_t^2}{c_l^2} - 2\Big)(r_2 - r_1), \qquad a_{12} = -\frac{i\omega}{\rho c_l^2}(r_2 - r_1),$$

$$a_{21} = i\rho\Big[\frac{4c_t^2}{\omega r^2}\Big(3 - 4\frac{c_t^2}{c_l^2}\Big) - \omega\Big](r_2 - r_1), \qquad a_{22} = -\frac{4c_t^2}{rc_l^2}(r_2 - r_1).$$

Let $\{\alpha(x), r_2\}$ be an optimal control from the admissible set (1.9) and (1.10) that minimizes functional (1.11) and satisfies constraint (1.12). We consider the perturbed control $\{\alpha^*(x), r_2 + \delta r_2\}$ [6]:

$$\alpha^*(x) = \begin{cases} v(x), & x \in D, \quad v(x) \in U, \\ \alpha(x), & x \notin D, \quad \operatorname{mes}(D) < \varepsilon, \end{cases} \qquad r_2 + \delta r_2 \in [a,\, b], \qquad |\delta r_2| < \varepsilon \tag{2.2}$$

($D \subset [0,\,1]$ is a set of small measure and $\varepsilon > 0$ is a small quantity). The variation (2.2) of the control $\{\alpha(x), r_2\}$ generates the variations δF_0 and δF_1 of functionals (1.11) and (1.12) (for brevity, the independent variables corresponding to the unperturbed control $\{\alpha(x), r_2\}$ are omitted):

$$\delta F_0 = \int\limits_D (G(\alpha^*,\ldots) - G(\alpha,\ldots))\,dx + S_0\,\delta r_2; \tag{2.3}$$

$$\delta F_1 = \delta J = \int\limits_D \Big[\int\limits_0^\infty \operatorname{Re}\,(M(\alpha^*,\ldots) - M(\alpha,\ldots))d\omega\Big]dx + S_1\,\delta r_2. \tag{2.4}$$

Here $M(\alpha, r_2, x, \omega, \boldsymbol{Y}, \boldsymbol{\Psi}) = \boldsymbol{\Psi}^{\mathrm{t}}(x,\omega)A(\alpha, r_2, x, \omega)\,\boldsymbol{Y}(x,\omega)$,

$$S_0 = \int\limits_0^1 \frac{\partial}{\partial r_2} G(\alpha, r_2, x)\,dx,$$

$$S_1 = \int\limits_0^\infty \Big\{\int\limits_0^1 \operatorname{Re}\Big(\frac{\partial}{\partial r_2} M(\alpha, r_2, x, \omega, \boldsymbol{Y}, \boldsymbol{\Psi})\Big)dx$$

$$+ 2\pi|y_2(1,\omega)|^2\operatorname{Re}\Big(\frac{\partial}{\partial r_2}\Big(r_2^2\,\frac{g_{22}}{g_{21}}\Big)\Big) + \operatorname{Re}\Big(\psi_1(1,\omega)y_2(1,\omega)\,\frac{\partial}{\partial r_2}\Big(\frac{g_{22}}{g_{21}}\Big)\Big)\Big\}d\omega.$$

For each fixed ω, the vector of the conjugate variables $\boldsymbol{\Psi}(x,\omega) = (\psi_1, \psi_2)^{\mathrm{t}}$ is determined from the solution of the boundary-value problem

$$\begin{gathered}\boldsymbol{\Psi}'(x,\omega) = -A^{\mathrm{t}}(\alpha, r_2, x, \omega)\boldsymbol{\Psi}(x,\omega), \qquad g_{12}\psi_1(0,\omega) - g_{11}\psi_2(0,\omega) = 0,\\ g_{22}\psi_1(1,\omega) - g_{21}\psi_2(1,\omega) = -4\pi r_2^2 g_{21}\bar{y}_2(1,\omega)\operatorname{Re}\,(g_{22}/g_{21}).\end{gathered} \tag{2.5}$$

We consider the extended functional

TABLE 1

Material	ρ, kg/m^3	c_l, m/sec	c_t, m/sec
Spheroplastic	650	2278	1279
Duralumin	2800	6129	3087
Titanium	4600	6110	3143
Steel	7800	6020	3218
Copper	8930	4394	2163
Lead	11340	1956	727
Rubber	930	72	17
Tin	7290	3188	1606
Glass	2400	5292	3055

$$Q(\alpha, r_2) = F_0(\alpha, r_2) + \lambda_1\{F_1(\alpha, r_2, \boldsymbol{Y}) + \xi_1^2\} + \lambda_2\{a - r_2 + \xi_2^2\} + \lambda_3\{r_2 - b + \xi_3^2\} \tag{2.6}$$

(λ_i and ξ_i are Lagrange multipliers and penalty variables, respectively). Using expressions (2.3) and (2.4), we write the variation of functional (2.6) in the form

$$\delta Q = \int\limits_D \{H(\alpha, \ldots) - H(\alpha^*, \ldots)\}\, dx + \{S_0 + \lambda_1 S_1 - \lambda_2 + \lambda_3\}\, \delta r_2 + 2\sum_{i=1}^{3} \xi_i \lambda_i\, \delta\xi_i, \tag{2.7}$$

where

$$H(\alpha, r_2, x, \boldsymbol{Y}, \boldsymbol{\Psi}) = -G(\alpha, r_2, x) - \lambda_1 \int\limits_0^\infty \operatorname{Re}\,(M(\alpha, r_2, x, \omega, \boldsymbol{Y}, \boldsymbol{\Psi}))\, d\omega. \tag{2.8}$$

Since the control $\{\alpha(x), r_2\}$ is optimal (minimizing), the condition $\delta Q \geqslant 0$ must hold for every admissible control $\{\alpha^*(x), r_2 + \delta r_2\}$ (2.2). From expression (2.7), by virtue of the arbitrariness of the variations δr_2 and $\delta\xi_i$, we obtain

$$S_0 + \lambda_1 S_1 - \lambda_2 + \lambda_3 = 0; \tag{2.9}$$

$$\lambda_1 F_1(\alpha, r_2, \boldsymbol{Y}) = 0, \qquad \lambda_1 \geqslant 0; \tag{2.10}$$

$$\lambda_2\,(a - r_2) = 0, \qquad \lambda_3\,(r_2 - b) = 0, \qquad \lambda_2 \geqslant 0, \qquad \lambda_3 \geqslant 0. \tag{2.11}$$

Since the small-measure set D can be closely arranged almost everywhere on the segment [0, 1], the condition of maximum of the Hamilton function H (2.8) for the argument α [6] must hold for almost all $x \in [0, 1]$

$$H(\alpha, r_2, x, \boldsymbol{Y}, \boldsymbol{\Psi}) = \max_{\alpha^*(x) \in U} H(\alpha^*, r_2, x, \boldsymbol{Y}, \boldsymbol{\Psi}). \tag{2.12}$$

Thus, the optimal control $\{\alpha(x), r_2\}$, the corresponding optimal trajectory $\boldsymbol{Y}\,(x, \omega)$, and the vector of the conjugate variables $\boldsymbol{\Psi}\,(x, \omega)$ must satisfy the boundary-value problems (2.1) and (2.5), relations (1.9), (1.10), (1.12), (2.10), and (2.11), and the optimality conditions (2.9) and (2.12).

The necessary optimality conditions obtained are used to develop an algorithm for the synthesis of a spherical screen [7].

3. Example of Calculation. The set W consists of nine materials, whose acoustic properties are listed in Table 1.

A monochromatic spherical wave with a frequency of $f = 10$ kHz ($\omega = 2\pi f$) is incident on the screen. The regions $r < r_1$ and $r > r_2$ are occupied by air: $\rho_a = \rho_b = 1.29$ kg/m^3, $c_{la} = c_{lb} = 331$ m/sec, and $c_{ta} = c_{tb} = 0$. The inner radius of the screen r_1 is assumed to be fixed and equal to 1 m, and the outer radius r_2 varies from 1.014 to 1.015 m. The transmission factor of the screen is $\eta = 10^{-8}$. The screen is divided across the thickness into 50 equal parts, which model small-measure sets on which the control is varied.

Calculations were performed with different initial approximations, which were chosen from numerical experiments. The result was a three-layer screen with outer radius $r_2 = 1.014687$ m and mass $F^* = 1762.938$ kg consisting of lead layers with thicknesses of 1–1.001175 and 1.012925–1.014687 m and a copper layer with a thickness of 1.001175–1.012925 m.

The lightest homogeneous screen that satisfies constraints (1.10) and (1.12) is a lead screen with outer radius $r_2 = 1.014$ m and mass $F_* = 2023.098$ kg.

The relative gain in mass for the optimal screen compared to the given homogeneous screen is $(1 - F^*/F_*) \cdot 100\% = 12.9\%$.

This example shows that the optimal structure includes materials of the highest density. Therefore, the gain in mass can be small. If the cost rather than the mass of the constituent materials is chosen as the objective function (1.13), the optimal structure can also consist of other materials.

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